Analytically Derived Wind-Wave Directional Spectrum*
Part 2. Characteristics, Comparison and Verification of Spectrum

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The dependence of the angular spreading on frequency and wind-wave growth status is discussed in great detail for the proposed spectrum. The calculated angular spreading agrees with the measurements of Donelan et al. but is slightly broader. Explanation is given to the appearance of the narrowest spreading at a frequency slightly smaller than that of the wind-wave frequency spectrum peak as found by these authors. There is also basic agreement between the calculated spreading and the formulas of Mitsuyasu et al. and Hasselmann et al. for the specific wind-wave status on which these empirical formulas are based, though the former is narrower. The wind-wave frequency spectrum obtained by integrating the proposed directional spectrum with respect to direction agrees with the JONSWAP spectrum and that derived by the authors previously. The proposed spectrum is preliminarily verified with field data obtained by optical method.

1. Introduction

In Part 1 of the present paper, the authors (Wen et al., 1993) derived analytically a wind-wave directional spectrum of the nondimensional form

\[
\tilde{F}(\tilde{\omega}, \theta) = \frac{k_1k_2}{k_3} P \cos^m \theta \tilde{\omega}^n \exp \left[ - \frac{P_0}{2} \left( \tilde{\omega}^n - 1 \right) \right], \quad \tilde{\omega} \leq \tilde{\omega}_i,
\]

\[
\tilde{F}(\tilde{\omega}, \theta) = \frac{\tilde{F}(\tilde{\omega}_i, \theta)}{\tilde{\omega}_i^4}, \quad \tilde{\omega} \geq \tilde{\omega}_i,
\]

where

\[
\tilde{F}(\tilde{\omega}, \theta) = \frac{\omega_0 F(\omega, \theta)}{m_0},
\]

\[
\tilde{\omega}_\theta = \frac{1}{k_2 \cos^2 \theta} \tilde{\omega},
\]

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\[ \tilde{\omega} = \omega / \omega_0, \]
\[ \tilde{\omega}_t = \omega_t / \omega_0, \]
\[ n = n_1 - n_2 + n_3, \]
\[ n_1 = 9.798 - 1.633P, \]
\[ n_2 = -0.20 + 0.0333P, \]
\[ n_3 = 1.80 - 0.30P, \]
\[ k_1 = 1.286 - 0.1599P + 0.0144P^2 - 0.0027P^3, \]
\[ k_2 = 0.990 + 0.001667P, \]
\[ k_3 = 1.047 + 0.0342P, \]
\[ p_\theta = e^{P_\theta} + 1, \]
\[ q_\theta = e^{P_\theta}, \]
\[ P_\theta = k_3 P \cos n_3 \theta, \]
\[ P = \omega_0 S(\omega_0)/m_0. \]

In these expressions, \( F(\omega, \theta) \) is the dimensional directional spectrum, \( \omega \) the angular frequency and \( \theta \) the direction measured from the dominant wind-wave direction; \( m_0, \omega_0 \) and \( P \) are respectively the zero order moment, peak frequency and peakness factor of the wind-wave frequency spectrum \( S(\omega) \); \( P_\theta \) is the peakness factor for the frequency spectrum in the direction \( \theta \); \( e \) is the base of natural logarithm; \( \omega_t \) is the lower limit frequency of the equilibrium range and the corresponding nondimensional frequency \( \tilde{\omega}_t \) is calculated from

\[ \tilde{\omega}_t = 2.38 P^{-0.406}. \]  

The spectrum in Eq. (1) differs greatly in form from existing formulas and contains \( m_0, \omega_0 \) and \( P \) as parameters. The parameter \( P \), as a measure of geometrical spectral width, changes with wind-wave growth and plays an important role in determining the spectral structure. We have in Part 1 of the paper discussed qualitatively the dependence of spectral form on wave development stage and the dependence of directionality on frequency. In the following these and other features of the proposed spectrum will quantitatively be elaborated. In many instances, our discussion is to be carried out for the cases of \( P = 1.538, 3 \) and 5, corresponding to fully developed, moderately developed and young wind waves respectively. As pointed out in Part 1 of the paper (Wen et al., 1993), the peakness factor \( P = 3 \) corresponds to the peak enhancement factor \( \gamma = 3.1 \) in the
JONSWAP spectrum for which the average value of $\gamma$ is 3.3. So we shall, as we did in Part 1 of the paper, take the case of $P = 3$ as representative of the wind-wave growth status which we ordinarily encounter in the sea.

We have given the relation (Wen et al., 1993)

$$P = 1.69 \frac{U\omega_0}{g},$$

(3)

where $U$ is the wind speed at 10 meter height above sea surface and $g$ the gravitational acceleration. The quantity $U\omega_0 / g$ actually represents the ratio between the wind speed and the phase speed of the component wave with the peak frequency $\omega_0$. Thus, the three wave development stages mentioned above correspond, by Eq. (3), to the speed ratios $U\omega_0 / g = 0.91, 1.78$ and 2.96 respectively.

We will compare the proposed spectrum with existing formulas based on measurements and verify it with some field data obtained by optical method.

2. Characteristics of the Proposed Directional Spectrum

2.1 Two-dimensional contour plot and three-dimensional shape of spectrum

Figure 1 shows the contour lines of spectral magnitude of Eq. (1) plotted on the $\omega$-$\theta$ plane, figures on the lines being values of $\tilde{F}(\tilde{\omega}, \theta)$ for $P = 1.538, 3$ and 5. It is interesting to note that in the course of wind-wave growth as specified by decreasing $P$ the angular distribution of spectral energy becomes narrower, while the distribution with respect to frequency widens for developed waves. These features of change of spectral form can be further demonstrated by the three-dimensional plot in Fig. 2, where the vertical axes represent the spectral magnitudes $\tilde{F}(\tilde{\omega}, \theta)$ of Eq. (1) and the horizontal axes constitute the $\tilde{\omega}$-$\theta$ planes. Peak values are indicated in the figure. Both Figs. 1 and 2 show the dependence of spectral structure on wave development stages.

2.2 Frequency spectra in different directions

The frequency spectra $\tilde{S}_\theta(\tilde{\omega})$ along the direction $\theta$ as computed from Eq. (1) for given $\theta$ are shown in Fig. 3 for $\theta = 0^\circ, 15^\circ, 30^\circ, 45^\circ$ and $60^\circ$ in the case of $P = 3$. The spectra become flatter when $\theta$ increases because of the decrease of $P_\theta$, the peakness factor of $\tilde{S}_\theta(\tilde{\omega})$. There is also a shifting of peak frequency $\tilde{\omega}_{0\theta}$ toward higher frequencies. But the shifting is so slight for small $\theta$ that it is hardly discernible in the figure, though the frequency would theoretically become infinite at $\theta = \pm \pi/2$. The area $\tilde{m}_{0\theta}$ under the spectral curve, a measure of energy, decreases slowly for small values of $\theta$ but rapidly for large ones. The rate of decrease of $\tilde{m}_{0\theta}$, however, decrease with increasing $P$. Thus, should curves similar to those in Fig. 3 be plotted, it would be seen that the energy beyond $\theta = 60^\circ$ is practically negligible for $P = 1.538$ while it is still of appreciable magnitude for $P = 5$. All these features are consistent with those manifested in Figs. 1 and 2.

2.3 Dependence of angular spreading on frequency

It has been confirmed by wave measurements that the angular spreading of spectral energy in a directional spectrum depends on frequency. The narrowest spreading appears approximately at the peak frequency and the farther is the frequency from that of the peak on both sides of the latter, the wider is the spreading. Such peculiarity of spectral energy distribution can be shown
Fig. 1. Contour lines of nondimensional magnitudes of the directional spectrum in Eq. (1). Figures (a), (b) and (c) refer to cases of $P = 1.538$, 3 and 5 (equivalent values of $U\omega_0/g = 0.91$, 1.78 and 2.96), corresponding to fully developed, moderately developed and young waves respectively.
Let $\tilde{F}(\tilde{\omega}, \theta)$ and $\tilde{F}(\tilde{\omega}, 0)$ be frequency spectra for direction $\theta$ and $\theta = 0$ respectively. Then the ratio

$$f(\tilde{\omega}, \theta) = \frac{\tilde{F}(\tilde{\omega}, \theta)}{\tilde{F}(\tilde{\omega}, 0)}$$

is a measure of spectral energy spreading, larger rate of decrease of $f(\tilde{\omega}, \theta)$ with respect to $\theta$ for small values of $\theta$ corresponding to more energy that is concentrated near $\theta = 0$. In fact, $f(\tilde{\omega}, \theta)$ is the very directional function which was first defined by Longuet-Higgins et al. (1963) and the function proposed by Donelan et al. (1985). Figure 4 shows the function $f(\tilde{\omega}, \theta)$ for $P = 3$. Curves are drawn for different frequency ratios $\tilde{\omega} = \omega/\omega_0$. The innermost curve corresponds to $\tilde{\omega} = 0.95$ at which the narrowest spreading makes its appearance. For frequencies both larger and smaller than 0.95, the spreading is broader. The outermost curve refers to the frequency representing the lower limit of equilibrium range beyond which the angular distribution of spectral energy is no longer dependent on frequency. Families of curves similar to that shown in Fig. 4 can be plotted for larger and smaller values than $P = 3$ and also exhibit the features just mentioned. Besides, younger waves, characterized by larger $P$, have wider spreading than the more developed.

Though the dependence of angular spreading on frequency was noted as early as in the sixties and confirmed in subsequent wave measurements and analyses, no attempt has been made to explain it on a physical ground. The dependence is, in fact, clear if we examine the family of frequency spectra in Fig. 3. Since $P_\theta$ decreases with increasing $\theta$ and the curves become flatter...
for larger $\theta$ and on both sides of $\tilde{\omega} = 1$, the ratio defined in Eq. (4) has its largest value near the peak frequency and smaller values at other frequencies, and this leads, in turn, to the narrowest angular spreading of energy in the vicinity of peak frequency and broader spreading for distant frequencies. Moreover, the more distant from the peak is the frequency, the broader is the spreading. This feature of energy spreading is just what is exhibited in Fig. 4. This peculiarity of the angular spreading can further be traced to the wind-wave generation process. Considering the effectiveness of energy transfer from wind to waves, the frequency spectrum in direction $\theta$ is under the action of smaller wind speed than that for $\theta = 0$ and the ratio between the effective speed of wind and the phase speed corresponding to the peak frequency is smaller in the former case. So the frequency spectrum in the direction $\theta$ is more developed and characterized by smaller $P_{\theta}$. Hence, the wider spreading for frequencies other than that at or near the peak.

Since the spectral curves in Eq. (1) become flatter and flatter as $\theta$ increases, in the frequency range $\tilde{\omega} = 0$ to 1 there will be a frequency below which the ratio computed from (4) is larger than unity. This, however, has no substantial effect on the application of the directional spectrum in Eq. (1), for at the frequency where the just mentioned erratic behavior of the ratio appears, the spectral energy is negligibly small.

Fig. 2. Three-dimensional plots of the directional spectrum in Eq. (1) for $P = 1.538$, 3 and 5 (equivalent values of $U_{0}\omega/\sqrt{g} = 0.91$, 1.78 and 2.96). Vertical axes represent the nondimensional spectral magnitudes with peak values marked in the figures. Dependence of angular energy spreading on wave growth stage is well illustrated in both Figs. 1 and 2.
2.4 The narrowest angular spreading

Donelan et al. (1985) in their careful measurements and analyses of directional spectra found that the narrowest angular spreading appears at $\tilde{\omega} = 0.95$ rather than at the peak frequency $\tilde{\omega} = 1$ of the wind-wave frequency spectrum as conceived in previous researches by other authors. We proceed to explain this interesting phenomenon by the structure of the directional spectrum in Eq. (1). In the discussion of Part 1, it has been noted that there must be a shifting of peak frequencies of the frequency spectra for different $\theta$ in a directional spectrum and this in turn requires that the peak frequency $\omega_0$ of frequency spectrum for $\theta = 0$ is slightly smaller than the peak frequency $\omega_0$ of the wind-wave spectrum as a whole. Since a spectrum has its maximum magnitude at its peak, it is to be expected that the function $f(\tilde{\omega}, \theta)$ defined in Eq. (4) attains its largest value near $\omega_0$ and, as a consequence, the narrowest spreading appears near this frequency.

Let $\tilde{\omega}_m$ be the frequency at which the function $f(\tilde{\omega}, \theta)$ in Eq. (4) has its minimum value for given $\theta$. Then, for the directional spectrum in Eq. (1) computed with frequency interval $\Delta \tilde{\omega} = 0.01$ and direction interval $\Delta \theta = 1^\circ$, interpolated values of $\tilde{\omega}_m$ are plotted with smoothed curves in Fig. 5 for $P = 1.538, 3$ and $5$. It is seen that all the curves start at $\tilde{\omega}_m = 0.93$ approximately, remain substantially constant at first and then increase with $\theta$. Values of $f(\tilde{\omega}, \theta)$ for large $\theta$ are
Fig. 3. Nondimensional frequency spectra $\tilde{S}_\theta(\tilde{\omega})$ in different directions for $\theta = 0^\circ, 15^\circ, 30^\circ, 45^\circ$ and $60^\circ$. As $\theta$ increases, the area under the curve decreases and the curve becomes flatter.
too small to be used for the identification of $\tilde{\omega} m$, so the curves stop at certain angles. Though $\tilde{\omega} m$ has for given $\theta$ larger values for smaller $P$, the curves for different $P$ lie close to each other. Thus, the computed frequency $\tilde{\omega} m$ is basically within a belt with $\tilde{\omega} m = 0.93$ as its lower boundary and, for angles within which the major part of spectral energy is concentrated, it is smaller than $\tilde{\omega} m = 0.95$. The frequency $\tilde{\omega} = 0.95$ at which the narrowest spreading occurs as found in the measurements of Donelan et al. (1985) is plotted with a horizontal broken line in Fig. 5. Curves in the figure indicates that the peculiarity of the narrowest spreading found by Donelan et al. is likely a consequence of the appearance of $\tilde{\omega} m$.

2.5 Angular spreading in the equilibrium range

Considering the balance among the wind input, dissipation caused by wave breaking and the wave-wave nonlinear interaction, Phillips (1985) derived for the equilibrium range of wind-wave spectrum a form of $\sim \omega^{-4}$ and showed that the angular spreading in this range is proportional to $\cos^2 \theta$, being independent of frequency. This result agrees with observations of Donelan et al. (1985). We shall discuss the problem in the frame of the directional spectrum in Eq. (1).

For the equilibrium range of the spectrum in Eq. (1), the spreading function $f(\tilde{\omega}, \theta)$ defined in Eq. (4) is evidently independent of frequency. The solid lines in Fig. 6 show the computed spreading functions for $P = 1.538$, 3 and 5, the corresponding lower limit frequencies being, by Eq. (2), $\tilde{\omega}_l = 2.0$, 1.52 and 1.24. We have compared the computed functions $f(\tilde{\omega}, \theta)$ with functions of the form $\cos^r \theta$ with various values of the exponent $r$. The results of taking $r = 4$, 2 and 1/2 are plotted with broken lines in Fig. 6. Since $f(\tilde{\omega}, \theta)$ and $\cos^r \theta$ are even functions, the solid and broken lines have been drawn only in the range of either $0 - \pi/2$ or $0 - -\pi/2$ to save space.
The two sets of curves are close to each other, especially for ranges of small $\pm \theta$ within which most of the spectral energy is concentrated. Similar results have been obtained for the cases of $P = 2$ and 4, the corresponding values of $r$ being 3 and 1 respectively. It is interesting to note that, in the case of $P = 3$, the spreading function $f(\tilde{\omega}, \theta)$ for the equilibrium range based on the directional spectrum in Eq. (1) is close to the function $\cos^2 \theta$ which coincides with Phillips’ (1985) theory and agrees with the measurements of Donelan et al. (1985). The coincidence and agreement are,
to a certain extent, what can be expected, for the quoted authors’ research has been directed to the average status of wind-wave development which may be represented by \( P = 3 \) in our studies.

### 3. Comparison between Proposed Spectrum and Formulas Based on Observations

#### 3.1 Comparison with the formula of Donelan et al.

We proceed to compare the angular spreading calculated from the spectrum in Eq. (1) with those given by existing formulas. Among the formulas based on fields measurements, that recently proposed by Donelan et al. (1985) is important for three reasons: (1) the data used for constructing their formula were obtained from an array consisting of as many as 14 sensors which enabled finer directionality of a spectrum to be computed; (2) improved procedures had been employed for spectrum estimates; and (3) the wind-wave data were collected in a take and thus not or much less contaminated by the presence of swell. The speed ratio \( U\omega_0 / g \), which is a measure of wave development stage, ranged from 0.83 to 4.6 in their measurements, covering virtually all stages of wave development. Though the dependence of angular spreading on \( U\omega_0 / g \) can be discerned in the quoted authors’ published results, for instance, in Figs. 29, 30 and 32 of their paper, this speed ratio does not enter into the proposed formula for the spreading. So we assume that the formula of Donelan et al. is to be applied to the ordinarily encountered, or the average wind-wave development status which we have described by the peakness factor \( P = 3 \), the corresponding value of \( U\omega_0/g \) from Eq. (3) being 1.78.

The directional function of Donelan et al. is

\[
f(\tilde{\omega}, \theta) = \text{sech}^2 \beta \theta, \tag{5}
\]

where

\[
\beta = \begin{cases} 
  2.61\tilde{\omega}^{1.3}, & 0.56 < \tilde{\omega} < 0.95, \\
  2.28\tilde{\omega}^{-1.3}, & 0.95 < \tilde{\omega} < 1.6, \\
  1.24, & \text{other values of } \tilde{\omega}.
\end{cases}
\]

The function \( f(\tilde{\omega}, \theta) \) calculated from Eq. (4) and the spectrum in Eq. (1) for \( \tilde{\omega} = 0.85, 0.95, 1.3 \) and 1.6 is plotted with solid lines in Fig. 7 while the broken lines show the corresponding function in Eq. (5). The curves for the above four frequencies are shown only on either the right or the left side of the vertical axis. The dependence of spreading on frequency is evident for both empirical curves and those based on Eq. (1) and has the features mentioned earlier.

#### 3.2 Comparison with the formula of Hasselmann et al.

The formula of Hasselmann et al. (1980) is based on data of the JONSWAP experiments. Specifically, it was constructed by regression analysis of the Data set A in their paper which consisted of 4 sub-sets with the following ranges of speed ratio \( U\omega_0/g \): 1.0–1.2, 1.2–1.4, 1.6–1.8 and 1.0–1.8. We first compute the mean values of these sub-sets and then obtain the overall mean value \( U\omega_0/g = 1.38 \), which, by Eq. (3), corresponds to \( P = 2.33 \). We shall make our comparison by using this value of \( P \) which represents a wind-wave status more developed than that discussed
above in connection with the formula of Donelan et al.

Hasselmann et al. (1980) adopted for the angular spreading a function which was first proposed by Longuet-Higgins et al. (1963) and has the form

$$f(\tilde{\omega}, \theta) = \cos^s_{\frac{1}{2}} \theta,$$

where the quantity $s$ in the exponent is a function of frequency. Hasselmann et al., through fitting the field data, gave for $s$ the expression

$$s = s_m \tilde{\omega}^\mu,$$

where, for $\tilde{\omega} \geq 1$,

$$s_m = 9.77, \quad \mu = -2.33 - 1.45 \left( \frac{U \omega_0}{g} - 1.17 \right),$$

and for $\tilde{\omega} < 1$,

$$s_m = 6.97, \quad \mu = 4.06.$$
These formulas apply to the case \( U \gtrapprox g/\omega_0 \) and the error terms that originally appeared in the expressions of \( s_m \) and \( \mu \) for designating the fitting accuracy have been omitted here.

Substituting \( U\omega_0/g = 1.38 \) in the expression for \( \mu \), the angular spreading given by Eqs. (6) and (7) for \( \tilde{\omega} = 0.85, 0.95, 1.15 \) and 1.5 are plotted with broken lines in Fig. 8 in which, as in Fig. 7, curves are only drawn on either the right or the left side of the vertical axis. The solid lines in the figure show the spreading computed from the spectrum in Eq. (1) with \( P = 2.33 \). The curves based on Eq. (1) have appreciably narrower spreading than those based on Eqs. (6) and (7), especially for frequencies which are farther from the peak. This feature can also be observed in Fig. 7 of the paper by Donelan \textit{et al.} (1985) in which the angular spreading given by different formulas, including that of Hasselmann \textit{et al.}, was compared with typical measured directional spectra.

\subsection*{3.3 Comparison with the formula of Mitsuyasu \textit{et al.}}

Mitsuyasu \textit{et al.} (1975) measured the wind-wave directional spectra with a clover-leaf buoy and obtained a formula for the angular spreading on the basis of 5 sets of data. The approximate speed ratios \( U\omega_0/g \) read from Fig. 12 of their paper for these data sets are 0.905, 1.0, 1.0, 1.05 and 0.715 with an average equal to 0.934. The last figure corresponds, by Eq. (3), to \( P = 1.538 \) which almost coincides with the criterion \( P = 1.538 \) Wen \textit{et al.} (1988) obtained for fully developed wind waves in deep water. In fact, we have calculated by deep water formula the phase speeds of the significant waves involved in the data sets with the wave periods given in Table 1 of the paper by Mitsuyasu \textit{et al.} The ratios between wind speed and phase speed obtained in this way average 1.04, which is very close to unity. So the wind-wave status to which the data sets correspond can indeed be considered as almost fully developed.

The function in Eq. (6) was also used by Mitsuyasu \textit{et al.} to represent the angular spreading
and the quantity \( s \) they gave can be rewritten in the form

\[
\begin{align*}
\dot{s} &= s_{m} \omega^{-2.5}, & \omega &\equiv 1, \\
\ddot{s} &= s_{m} \omega^{-5}, & \omega &\equiv 1,
\end{align*}
\] (8)

where

\[
\dot{s}_{m} = 11.5 \left( \frac{U\omega_{0}}{g} \right)^{-2.5}.
\] (9)

The solid lines in Fig. 9 show the angular spreading calculated from the spectrum in Eq. (1) with \( P = 1.58 \) substituted in it. Since the wind-wave frequency spectrum has its largest width in the present case, the computation has been done for 4 frequencies ranging from \( \omega = 0.75 \) to 2.0. The spreading given by Eq. (8) with \( U\omega_{0}/g = 0.934 \) is plotted with broken lines, each of which lies above the corresponding solid one. While there is better agreement between the two methods for \( \omega = 0.95 \), the difference between them becomes remarkable for other frequencies. It seems that though the angular spreading computed from the formula of Mitsuyasu \textit{et al.} is narrow enough near the peak, it widens too rapidly at both high and low frequencies.

3.4 Comparison of the four methods for \( P = 3 \) and \( \omega = 1 \)

The angular spreading of spectral energy depends on the wind-wave status as well as

![Fig. 9. Comparison of directional function \( f(\omega, \theta) \) based on Eq. (1) with the formula of Mitsuyasu \textit{et al.} (1975), the former being plotted with solid lines computed for \( P = 1.58 \) and the latter with broken lines computed for \( U\omega_{0}/g = 0.934 \). These two quantities are equivalent by Eq. (3). Curves for \( \omega = 0.75, 0.95, 1.5 \) and 2.0 are shown only in the range \( \theta = -\pi/2 \) to \( 0 \) or \( 0 \) to \( \pi/2 \).]
frequency. In the above sub-sections, we have compared the spreading based on the directional spectrum in Eq. (1) with formulas based on data collected at average speed ratios $U/\omega_0 = 1.78$, 1.38 and 0.934, the corresponding peakness factor $P$ being 3, 2.33 and 1.58. There is no such formula that can be used for comparison with younger waves characterized by higher values of $U/\omega_0$ or $P$. Furthermore, the comparison with each formula has to be made separately, for different formulas apply generally to the specific ranges of $U/\omega_0$ or $P$ for which the formulas have been constructed.

However, it would be interesting to compare the different methods we discussed above for calculating the directional spreading under the same wave generation condition. To a certain extent, this is possible with the comparison to be made for $P = 3$ and $\tilde{\omega} = 1$, for, as explained before, Eq. (5) can be considered as representing the case of average status of wave development described by $P = 3$ and the quantity $s$ in Eq. (7) is independent of $U/\omega_0$ or $P$ when $\tilde{\omega} = 1$, meaning that the quantity $s$ obtained in this special case applies to wave status $P = 3$ as well. As for Eq. (8), we shall, for the time being, assume that it applies to the case of $P = 3$, though it was obtained from data in which the average value of $U/\omega_0$ is 0.934 corresponding to $P = 1.58$.

For $P = 3$ and $\tilde{\omega} = 1$, the spreading function for the spectrum in Eq. (1) reduces to the simple form

$$f_1(\tilde{\omega}, \theta) = \cos^{5.9} \theta,$$

while those from Eqs. (5), (7) and (8) are respectively

$$f_2(\tilde{\omega}, \theta) = \text{sech}^2 2.28 \theta, \quad \text{(Donelan et al.)}$$

$$f_3(\tilde{\omega}, \theta) = \left( \cos \frac{1}{2} \theta \right)^{19.5}, \quad \text{(Hasselmann et al.)}$$

$$f_4(\tilde{\omega}, \theta) = \left( \cos \frac{1}{2} \theta \right)^{5.48}. \quad \text{(Mitsuyasu et al.)}$$

The functions in Eqs. (10) and (11) are plotted in Fig. 10. The angular spreading computed with the first three functions are close to each other with the curve in Eq. (10) lying between the rest of them. The function $f_1(\tilde{\omega}, \theta)$ shows much wider spreading. This is in fact what can be expected, for the formula of Mitsuyasu et al. was not originally constructed from data of wave status represented by $P = 3$ or $U/\omega_0 = 1.78$. Should $U/\omega_0 = 0.934$, on which Eq. (8) is based, and $\tilde{\omega} = 1$ be substituted in Eqs. (8) and (9) the exponent on the right-hand side of $f_1(\tilde{\omega}, \theta)$ would be 27.2. Then the resulted spreading would be even narrower than that given by $f_1(\tilde{\omega}, \theta)$ in Eq. (11) and very close to that given by $f_1(\tilde{\omega}, \theta)$ in (10). This emphasizes the effect of the speed ratio $U/\omega_0$ in using the formula of Mitsuyasu et al. and may also have some significance in using other formulas.

The reason for our having chosen the case of $P = 3$ and $\tilde{\omega} = 1$ in the above comparison is twofold: the case of $P = 3$ represents the frequently encountered wind-wave status and the spectral energy is concentrated near the peak frequency $\tilde{\omega} = 1$. So the spreading functions in Eqs.
(10) and (11) obtained on the basis we choose would tell us the basic features of the various formulas so far proposed for calculating the angular spreading in the energy containing part of a wind-wave spectrum and show in what degree the spreading computed by the directional spectrum in Eq. (1) agrees with these formulas based on field measurements.

3.5 Comparison with spreading function of the form $\cos^n \theta$

Though the angular spreading depends on frequency and wave growth status as discussed in the foregoing Sections, in many theoretical and applied studies of ocean waves since the early fifties spreading function of the form

$$f(\theta) = K \cos^n \theta$$  \hspace{1cm} (12)$$

has been used, here $K$ and $n$ being constants. Pierson et al. (1955) in their method of wave prediction proposed the spreading function $(2/\pi)\cos^2 \theta$ which is widely employed even at the present day probably because of its simplicity. Other authors found from field measurements various values for $n$; Krelov (1966), for instance, gave $n = 3$. Barnett (1968) and Ewing (1971) in their numerical models of wave prediction took $n = 4$.

Despite the lacking of adequate experimental and theoretical support for the function in (12), its wide usage seems to warrant a comparison between it and other functions we have discussed. Since systematical comparisons have been made between the angular spreading of the spectrum in Eq. (1) and those represented by (5), (6) and (8), we shall limit ourselves to comparing the spectrum based on (1) with that given by (12).

We have compared the angular spreading for specific frequencies. But this is impossible for Eq. (12), for the spreading it gives is independent of frequency. So we shall compare the total

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Fig. 10. Comparison of directional function $f(\tilde{\omega}, \theta)$ based on Eq. (1) with the formulas of Donelan et al. (1985), Hasselmann et al. (1980) and Mitsuyasu et al. (1975) for $\tilde{\omega} = 1$, the parameter $p = 3$ being used in Eq. (1).
energy contributed by all frequencies within different direction ranges. For this purpose, we compute the ratio $R$ between the energy contained within $\pm \theta$ and the total energy of the spectrum by substituting the directional spectrum of Eq. (1) in

$$R = \frac{\int_{-\pi/2}^{\pi/2} \tilde{F}(\tilde{\omega}, \theta; P) \tilde{\omega} d\tilde{\omega} d\theta}{\int_{-\pi/2}^{\pi/2} \tilde{F}(\tilde{\omega}, \theta; P) d\tilde{\omega} d\theta} \tag{13}$$

When the integrating in (13) is done numerically in the frequency range $\tilde{\omega} = 0$–6, the ratios $R$ for $P = 1.538, 3$ and 5 are shown with solid lines in Fig. 11. The ratios corresponding to the angular spreading in (12) can be computed from

$$R = \frac{\int_{-\pi/2}^{\pi/2} \cos^n \theta d\theta}{\int_{-\pi/2}^{\pi/2} \cos^n \theta d\theta} \tag{14}$$

The broken lines in the figure show the ratios for $n = 2, 4$ and 6. It is interesting to note that this latter set of broken curves lie close to solid ones for $P = 5, 3$ and 1.538 respectively. Thus from the point of view of energy concentration as represented by the ratio $R$, the exponent $n$ in the spreading function (12) should change with wind-wave growth stage and, roughly speaking, the cases of $n = 2, 4$ and 6 correspond to very young, moderately developed and well developed waves respectively. This casts much doubt on the validity of the wide use of $\cos 2\theta$ as a spreading function in directional spectrum. For fully developed waves, frequently encountered waves and

![Fig. 11. Comparison of the ratio $R$ in Eq. (13) with that in Eq. (14), curves being plotted with solid and broken lines respectively. In Eq. (13) $P = 1.538, 3$ and 5 are used while in Eq. (14) we take $n = 6, 4$ and 2.](image-url)
the initial stage of wind-wave development as respectively represented by $P = 1.538, 3$ and $5$, the exponent $n$ in (12) has the corresponding approximate values $7, 5$ and $3/2$ based on estimate from the curves in Fig. 11.

The reasons for narrower angular spreading of spectral energy for more developed waves as characterized by smaller $P$ can be traced to two facts: (1) the rate of decrease of total energy

![Graph](image)

**Fig. 12.** Comparison of the wind-wave frequency spectra (solid lines) obtained from the directional spectrum in Eq. (1) with the JONSWAP spectra (broken lines) and the spectra (dotted lines) proposed by Wen et al. (1989). Figures (a), (b) and (c) show the spectra for $P = 1.538, 3$ and $5$ respectively, various stages of wind-wave growth being covered. The peak enhancement factor $\gamma$ in JONSWAP spectrum is obtained from a relation between $P$ and $\gamma$ given in Part 1 of the present paper (Wen et al., 1993).
represented by \( m_{00} \) in this case with respect to \( \theta \) is larger as manifested by curves like those in Fig. 3; (2) in more developed waves the angular spreading for all frequencies is narrower as manifested by curves like those in Fig. 4.

### 3.6 Comparison of wind-wave frequency spectra

As pointed out at the beginning of the Part 1 of the paper, the various wind-wave directional spectra so far proposed have been constructed by multiplying the frequency spectrum by a directional function obtained from measurements. In contrast with this approach, the directional spectrum in Eq. (1) was derived analytically without considering beforehand the form of the wind-wave frequency spectrum. Here is an interesting and important question: if Eq. (1) is integrated with respect to \( \theta \), how the resulted frequency spectrum looks like.

The solid lines in Fig. 12 show the frequency spectra obtained by integrating Eq. (1) with respect to \( \theta \) for the cases of \( P = 1.538, 3 \) and 5, the vertical and horizontal axes representing the nondimensional spectrum \( \omega S(\omega) / m_0 \) and frequency \( \tilde{\omega} \) respectively. The JONSWAP spectra are plotted in the figure with broken lines, corresponding values of the peak enhancement factor \( \gamma \) being computed from a relation between \( P \) and \( \gamma \) given in Part 1 of the paper (Wen et al., 1993). Plotted also in the figure with dotted lines are the improved forms of spectra of Wen et al. (1989) with \( m_0, \omega_0 \) and \( P \) as parameters.

Though the validity of the JONSWAP spectrum has been confirmed by wave data from wide and different sources, it fails for very large \( \gamma \) to describe well the rapid rise of the spectral form in the low frequency range because of the presence of \( \omega^4 \) in the expression of the spectrum. On the other hand, the rear face of the spectrum obtained from Eq. (1) rises too rapidly for large \( P \). The difference between the two spectra in this respect can easily be discerned in Fig. 12. The frequency spectrum of Wen et al. (1989) had been compared with data collected from various sea regions of China with satisfactory agreement. It will be noted that, despite the minor differences on the higher and lower frequency sides of the peak, the three spectra under
consideration are close to each other in the energy-containing part of the spectrum, especially in the case of fully developed waves characterized by $P = 1.538$.

4. Preliminary Verification of Directional Spectrum with Data Obtained through Optical Method

The directional spreading obtained from the spectrum in Eq. (1) was in Section 3 compared with existing formulas and so long as the comparison was made on the basis of similar wave growth status, the agreement may be considered as acceptable to a certain extent. Since the existing formulas had been constructed by fitting data from buoys or wave staff array, the spectrum in Eq. (1) was in fact verified by such means of measurement in an indirect way. We proceed to make preliminary verification of the spectrum by using data obtained through optical method.

Stilwell (1969) proposed a method to obtain ocean wave directional spectrum from sea surface photographs. The method was based on a first order theory of the irradiance at the camera and the theory was subsequently refined by Kasevich et al. (1972). These authors designed an optical system and measured the directional spectrum from the output film by using a densitometer. Scientists of the Ocean Optics, Remote Sensing and Information Processing (OORSIP) Lab., affiliated with the Ocean University of Qingdao, Qingdao, China, recently applied Stilwell’s idea to the measurement of directional spectrum with an improved optical system and processed the optical density data stored in the output film by computers.

Aerial photographs were taken with a camera of 400 mm focus length installed on an IL-14 plane. Two flights were flown in 1987 and 1988 over the Bohai Sea and the East China Sea at a height of about 2000 meters and a speed of 300 Km/hr. More than 300 photographs were obtained under the conditions of clear sky, vertical optical axis and small sea waves so that the requirements for applying the first order theory could be fulfilled. Each of the photographs was analyzed with the above mentioned optical/computer hybrid processing procedures and the

![Fig. 13. Verification of the nondimensional wave number spectrum $\tilde{S}(\tilde{k})$ (solid line) obtained by integration of Eq. (1) with data (shown by dots) of field measurements by optical method. The JONSWAP spectrum expressed in wave number is also plotted with broken line for comparison. The computed spectra correspond to $P = 2.56$.](image-url)
nondimensional directional spectrum computed with direction interval of $1^\circ$ and nondimensional wave number interval of 0.01266. Final report is in preparation by the OORSIP Lab. Here we limit ourselves to the result of such computation for one photograph and compare it with the spectrum in Eq. (1). The photograph was taken in the Bohai Sea with sea surface wind speed of 5 m/sec and direction of WNW. The fetch and significant wave height were about 50 km and 0.4 m respectively. The peakness factor $P$ of the wind-wave frequency spectrum under consideration is 2.56. Since the Bohai Sea is semiclosed, the effect of swell could be neglected on the observation site.

The measured spectral values were given for different wave numbers $k$ and directions $\theta$, where, for deep water, $k = \omega^2/g$. These values have appreciable fluctuations which need to be smoothed. For this purpose, we take the spectral magnitude at wave number $k$ to be equal to the average value.

Fig. 14. Nondimensional wave number spectra obtained from Eq. (1) for different directions and the corresponding measured results. The solid curves in figures (a), (b), (c) and (d) show the computed spectra for $\theta = 0^\circ$, $10^\circ$, $20^\circ$ and $40^\circ$ respectively and the dots represent the measured values obtained by optical means. Computations are made for $P = 2.56$. 
of the spectral magnitudes at 7 neighboring wave numbers with \(k\) at the center. Normalizing the measured spectrum by the peak wave number \(k_0\) and the zero order moment \(m_0\) and integrating the nondimensional directional spectrum \(\tilde{F}(\tilde{k}, \theta)\) with respect to \(\theta\) by numerical procedure, we get the nondimensional wave number spectrum \(\tilde{S}(\tilde{k}) = k_0 S(k)/m_0\) plotted against the nondimensional wave number \(\tilde{k} = k/k_0\) as shown by the dots in Fig. 13. Plotted also in the figure with solid line is the spectrum obtained by transforming Eq. (1) into wave number spectrum and integrating it with respect to \(\theta\) for the case of \(P = 2.56\). The broken line shows the JONSWAP spectrum expressed as function of wave number, the peak enhancement factor \(\gamma\) corresponding to \(P = 2.56\) being 2.35. It is seen that within the wave number range \(0.5 \leq k \leq 2.0\), the spectrum from Eq. (1) agrees well with observations, but beyond \(\tilde{k} > 2\), the measured spectral values lie appreciably below those given by Eq. (1) and the JONSWAP spectrum. The discrepancy is

Fig. 15. Dependence of angular energy spreading on wave number. The solid lines are computed from Eq. (1) for \(\tilde{k} = 0.73, 1.0, 1.7\) and 2.3, the directional function \(f(\theta)\) being obtained from Eq. (4) for given wave numbers. Results of optical measurements are plotted with dots for comparison.
probably caused by the failure of optical method to reflect the spectral structure in the high wave number or frequency range for some reasons to be explored.

Figure 14 shows the nondimensional wave number spectra $\tilde{S}(\tilde{k}) = k_0S(k)/m_0$ plotted against $\tilde{k}$ for $\theta = 0^\circ, 10^\circ, 20^\circ$ and $40^\circ$, the solid lines and dots representing respectively the spectra obtained from Eq. (1) and measured values. The agreement between computation and observations is satisfactory for the energy containing part near the peak up to $\theta = 20^\circ$, though the field measurements show higher peak values and lower spectral magnitudes for higher wave numbers. In the case of $\theta = 40^\circ$, the measured values become appreciably lower. The discrepancy may be due to the defects inherent in either Eq. (1) or the optical method employed in the measurements.

We next compare the dependence of angular spreading on wave number. The solid lines in Fig. 15 show the directional functions $f(\theta)$ calculated from spectrum in Eq. (1) for $\tilde{k} = 0.73, 1.0, 1.7$ and $2.3$ while the dots represent the corresponding measured values, $f(\theta)$ being obtained from Eq. (4) for given wave numbers. Computation agrees with observations for wave numbers adjacent to that of the peak. But, for $\tilde{k} \gg 1.7$, the measured spreading is systematically narrower, and its deviation from that based on Eq. (1) would be very large for still larger $\tilde{k}$. However, careful examination of the figure shows that, beginning $\tilde{k} = 1.0$, the measured spreading remains essentially unchanged for increasing $\tilde{k}$, the spreading for $\tilde{k} = 2.3$ being, in fact, even narrower than that at the peak. This is contrary to most observations and theoretical consideration and may perhaps explain the difference between the measured and computed spreading shown in Fig. 15 for larger values of $\tilde{k}$.

5. Conclusions

(1) The angular spreading of energy in a directional spectrum depends on the wind-wave growth stages. This is well illustrated by the 2- and 3-dimensional plots in Figs. 1 and 2. The more developed are the wind waves as characterized by smaller peakness factor $P$, or, equivalently, the speed ratio $U\omega/\sqrt{g}$, the narrower is the spreading.

(2) The dependence of angular spreading on frequency as represented by the directional function $f(\tilde{\omega}, \theta)$ in Eq. (4) can be explained by the different wave development status of the frequency spectra for different directions in a directional spectrum.

(3) The very slow shifting of peak frequencies among the frequency spectra in different directions leads to the appearing of the narrowest angular spreading at some frequencies close to $\tilde{\omega} = 0.95$, a figure found by Donelan et al. in field measurements.

(4) For the equilibrium range of wind-wave spectrum, the angular spreading tends to be independent of frequency but changes with wave growth stage, developed waves having narrower spreading. The spreading calculated from the directional spectrum in Eq. (1) can approximately be represented by function of the form $\cos^r \theta$, where the exponent $r$ varies with the peakness factor $P$. In the cases of fully developed waves ($P = 1.538$), moderately developed waves ($P = 3$) and very young waves ($P = 5$), the exponent $r$ may be taken equal to 4, 2 and $1/2$ respectively.

(5) Calculation of angular spreading of spectral energy shows the basic agreement between results based on Eq. (1) and the directional functions proposed by Mitsuyasu et al., Hasselmann et al. and Donelan et al. provided the comparison is made for the same wind-wave growth status. This also indicates the necessity of taking wind-wave states into account when these directional functions are to be applied.

(6) The common usage of $\cos^2 \theta$ as the spreading function in wind-wave directional
spectrum should be viewed critically. Aside from its incapability to reflect the dependence of spreading on frequency, this spreading function will give an angular distribution of total energy (as represented by the zero order moment of the frequency spectrum in the direction $\theta$) which applies only to young waves. Should the total energy distribution be expressed in the form $\cos^n\theta$, the exponent $n$ would change with wave growth status specified by the peakness factor $P$.

(7) The energy containing part of the frequency spectrum obtained by integrating the directional spectrum in Eq. (1) with respect to $\theta$ agrees with those of the JONSWAP spectrum and the wind-wave frequency spectrum proposed by the authors in 1989.

(8) The result of preliminary verification of the spectrum in Eq. (1) through optical measurement data is encouraging: there is basic agreement between computation and observations, but the discrepancy at high wave numbers or frequencies needs to be further explored.

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References


