Short Contribution

Cyclostrophic Balance in Surface Gravity Waves: Essay on Coriolis Effects

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When gravity waves of small amplitude progress over the surface of deep water, the particle orbits are observed to be closed circles; therefore the waves possess orbital angular momentum. The particles experience a balance of two equal and opposite forces, called the cyclostrophic balance: the outward centrifugal force and an inward pressure force. On the Earth’s surface the Coriolis force causes a minor disruption of the normal cyclostrophic balance. For plane waves it is possible that the Coriolis force can be balanced at all times, and the cyclostrophic balance can be maintained as well, by a slight change in the orientation and the shape of the particle orbits. In order to balance all three forces simultaneously in either hemisphere, the shape of the orbits should be oval (perhaps elliptical) in general, where the shorter axis of the oval is parallel to the mean surface, and the plane of the orbits should be tilted to the left of the direction of wave propagation. In the northern hemisphere the orbital planes should also be tilted to the left of the vertical and in the southern hemisphere to the right of the vertical, facing in the direction of wave propagation. For ocean swell the order of magnitude of the sine of both tilt angles, as well as the eccentricity of the orbits, is comparable to the ratio of the Coriolis parameter to wave frequency, or one in ten thousand, which is probably too small to be observed. If in some particular circumstance (perhaps transient conditions) the Coriolis force is not balanced, then a Coriolis torque exists that will try to change the direction, but not the magnitude, of the orbital angular momentum of the waves. The general form of the Coriolis torque is worked out in Appendix.

1. Introduction

When gravity waves of small amplitude progress over the surface of deep water, each fluid particle, within the depth of wave influence, is observed to move in a circular orbit about a fixed mean position. In other words the waves have orbital angular momentum. No matter which way surface gravity waves travel in the ocean, the fluid particles have this circular orbital velocity, and the Coriolis force operates on any velocity relative to the Earth’s surface. What effect does the Coriolis force have on the orbital velocity of the particle or on the angular momentum of the waves? An answer to this question will be given in the following brief report.

The discussion of the influence of the Coriolis force on surface gravity waves has had a moderately long history (about 50 years) but the list of references on the topic is quite short. Ursell (1950) raised a puzzle, or a paradox, that may not have been solved in an entirely satisfactory way even to this day. At finite amplitude observations show that the particle orbits are not quite closed circles: at the end of each wave period the particles are displaced slightly in the direction of wave propagation, the net result being a slow (Stokes) drift velocity or a linear momentum of the particles. On this small steady drift velocity the Coriolis force must be acting, for wave propagation over the Earth’s surface, and there is apparently no conceivable force to balance it. So what happens? Can the Stokes drift exist in the ocean or not? (To my knowledge measurements have not yet unambiguously revealed the presence of the Stokes drift in the ocean.) Hasselmann (1970) used the opportunity to come up with a theory for the generation of inertial motion by surface gravity waves.

In an application to a completely different (much larger) space/time scale Backus (1962) made a calculation of the influence of the Earth’s rotation on surface wave propagation over global distances. The computed effects he found were too small to be detected by the available measurements.
But I am not aware of any previous studies that deal particularly with the influence of the Coriolis force on the particle velocity of surface gravity waves during the complete orbital motion. The working of the Coriolis force on the orbiting fluid particles has received scant attention in the past. In fact, in the vast majority of theoretical works on surface gravity waves the Coriolis force is neglected entirely, and the assumptions behind the omission are usually not mentioned.

A traditional approximation used in oceanography for large-scale motions is to neglect the component of the Coriolis force that acts on the vertical velocity, or more generally to neglect the component of the Earth’s rotation that is locally parallel to the mean surface of the ocean (Eckart, 1960, p. 36). However this approximation cannot be made for surface gravity waves in most cases because the magnitude of the vertical velocity equals that of the horizontal velocity for the circularly orbiting fluid particles, and the horizontal and vertical components of the Earth’s rotation are comparable at most latitudes (except on or near the equator). Therefore both components of the Earth’s angular velocity will be taken into consideration here.

One could just as easily extend Ursell’s puzzle to the whole rest of particle’s orbit (or to the whole orbit in linear waves) and not restrict it to the tiny piece of the orbit that does not close in finite amplitude waves. And in fact since the particle velocity is much larger than the Stokes drift velocity, the Coriolis force is much larger for the particle velocity than it is for the Stokes velocity.

Is it possible that the Coriolis force on a fluid particle could be balanced at all times during the orbital motion, but we need to understand how such a balance can be achieved. This consideration is taken up in the main part of the text. If for some reason, such as might happen under transient conditions, the Coriolis force cannot be balanced, then there would exist a “Coriolis torque” that would cause a change in the orbital angular momentum of the waves. Therefore we need to understand this process as well, and the details of it are worked out in Appendix.

2.  Cyclostrophic Balance

Take the particle orbits to be circular and for the moment neglect the Coriolis force. At all times as a particle moves around in its orbit there is a balance of two equal but opposite forces, and this balance is called the cyclostrophic balance (Kenyon, 1991). The outward directed force is the centrifugal force and the inward force is a pressure force that is related to the variable height of the wave surface and to the vertical acceleration of the particles. The Coriolis force will disrupt this balance of forces, but the disruption will be a minor one as will be seen shortly. A balance among the three forces can be obtained by a slight adjustment in the orientation of the orbital plane and in the shape of the orbit.

First, to compare the magnitudes of the centrifugal and Coriolis forces on the orbital velocity for typical ocean wave conditions. Considering for the moment only the vertical component of the Earth’s rotation, the ratio of the centrifugal to the Coriolis force is

$$\frac{\text{centrifugal}}{\text{Coriolis}} = \frac{\omega}{f} = 10^4$$

where $f$ is the Coriolis parameter (twice the angular velocity of the Earth times the sine of the latitude) and $\omega$ is the wave frequency. For swell with periods of order 10 sec the ratio of the two forces at mid-latitudes has the order of magnitude of 10,000, because $f$ itself is order $10^{-4}$ sec$^{-1}$. The higher the frequency the larger the ratio (1) is. If the horizontal component of the Earth’s rotation is also taken into account, the ratio of forces will have the same order of magnitude as just stated. From this fact alone a logical guess can be made that it would only take a very small rearrangement of the orbital parameters of the fluid particle in order to keep the main balance of forces (the cyclostrophic one) in tact and to balance the Coriolis force at the same time.

3. Coriolis Effects

Now the Coriolis force will be brought directly into the picture and the assumption will be retained temporarily that the only component of the Earth’s rotation that is operating is the one locally perpendicular to the mean surface (or as an equivalent idealization one could think of waves propagating at or near the north pole). The vertical component of rotation produces a Coriolis force that acts exclusively on the horizontal component of the particle velocity.

Let the wave travel in the positive $y$-direction, which can have any orientation in the horizontal plane tangent to the Earth’s surface. The $z$-axis points up anti-parallel to gravity. Figure 1 gives the orbit of a surface particle in the vertical ($xz$-)plane, edge on so to speak. The view in the $yz$-plane (not shown) would portray a circle of radius equal to the wave amplitude, but the Coriolis effect could not be seen in such a view. Without the Coriolis force Fig. 1 would show a vertical line of length equal to twice the wave amplitude, which is the projection of the circular orbit onto this plane. In order that the Coriolis force can be balanced there must be a slight tilt of the vertical line to the left in the northern hemisphere, where the Coriolis force acts to the right of the velocity. The amount of tilt is greatly exaggerated in Fig. 1, for ease of illustration, as is the size of the Coriolis force in relation to the centrifugal force.

How the balance of forces can occur is indicated in Fig. 1. The three forces are shown for a particle at the surface of a crest. At the top of the tilted line segment the particle velocity is horizontal and therefore directed into the paper. When the orbit is tilted, the outward centrifugal force no longer acts vertically but there is a slight horizontal com-
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of the direction of wave propagation, and the amount of tilt is adjusted so that the Coriolis force is balanced. The major balance of forces in the wave direction, the cyclostrophic balance, is also still in effect.

Assume now that the wave travels due east (a wave traveling due north will not experience the following effect). At the trailing edge of the crest where the particle velocity is vertically downward the Coriolis force, due to the horizontal component of the Earth’s rotation, acts to the right of the horizontal particle velocity in the northern hemisphere. The vertical component of the centrifugal force $F_{ce} \cos \alpha$ balances the downward pressure force $F_p$, and this is the cyclostrophic balance.

As the second example consider only the horizontal component of the Earth’s rotation, or think of waves traveling at the equator. The Coriolis force associated with the horizontal component of the Earth’s rotation only acts on the vertical component of the particle velocity. Figure 2 shows the balance of forces at the trailing edge of the crest where the particle velocity is downward. For this purpose the horizontal plan view is selected (the $xy$-plane), and the wave travels in the positive $y$-direction, which to start with is assumed to be due north. At the top end of the tilted line segment the particle velocity is down and therefore into the paper. The slight tilt (angle $\beta$) of the orbital plane is to the left of the direction of wave propagation, and the amount of tilt is adjusted so that the Coriolis force is balanced. The major balance of forces in the wave direction, the cyclostrophic balance, is also still in effect.

Assume now that the wave travels due east (a wave traveling due north will not experience the following effect). At the trailing edge of the crest where the particle velocity is vertically downward the Coriolis force, due to the horizontal component of the Earth’s rotation, points in the direction of wave propagation, which is in the direction of the centrifugal force and in the opposite direction to the pressure force. Now due to the colinearity of the three forces there is no possibility of balancing the Coriolis force by a change in the orientation of the orbital plane. However, by a slight change in the shape of the orbit, the Coriolis force could be balanced. What is needed is a little smaller centrifugal force, because the given pressure force has to balance...
both the Coriolis and centrifugal forces. This could be accomplished by an oval or elliptical orbit, where the shorter axis of the ellipse is parallel to the mean surface and the longer axis remains equal twice the wave amplitude, for the surface particles. Then the radius of curvature would be larger, at the ends of the short axis, than for a circle, which would give a smaller centrifugal force. The eccentricity of such an ellipse would be so close to one, due to the small ratio of Coriolis to centrifugal forces, that it probably would not be measurable. (In this brief report an attempt is not made to prove mathematically whether or not the orbits are indeed ellipses.)

If for some reason the Coriolis force is not balanced (e.g. the orbital planes are not tilted), then each fluid particle will try to move (accelerate) in the direction of the Coriolis force. In the first example for waves traveling in any direction near the north pole, the particles at the crest would try to move to the right of the direction of wave propagation and to the left at the trough. The net result at both crest and trough would be a torque pointing in the direction of wave travel (see Appendix). Such a torque would attempt to change the direction of the angular momentum without altering its magnitude. In the last example of the wave traveling east at the equator, if the particle orbits remain circular so that the Coriolis force is not balanced, then a particle will try to move in the direction of wave propagation on the back face of the wave and in the opposite direction on the front face. Again the result will be a torque that tries to change the direction of the angular momentum vector.

By the nature of the way a torque operates on the angular momentum, if the Coriolis force is not initially balanced by another force, then the torque caused by the Coriolis force will not tend to bring it into balance because the torque will only try to change the direction of the angular momentum. A change in the direction of the angular momentum will leave the Coriolis force still unbalanced. (One could think of the classical problem of a spinning bicycle wheel supported by a point on its axis that precesses in the horizontal plane due to the torque caused by the vertical force of gravity.)

4. Discussion

For wave propagation at an arbitrary latitude in the northern hemisphere, and assuming all the forces are balanced, both types of tilts of the orbital plane shown in Figs. 1 and 2 will occur simultaneously (it may be difficult to visualize the two effects in a single diagram, however). Of course the tilt angles are very small and probably not measurable: for example \( \sin \alpha = \alpha = \beta/\omega = 10^{-4} \), and the angle \( \beta \) will have a similar order of magnitude. In addition when the waves travel in an arbitrary direction (except due north) over the ocean surface, and all the forces are balanced, then one expects the shape of the particle orbits to be slightly elliptical in either hemisphere. The type of ellipticity caused by the Coriolis force is qualitatively different from that observed when waves shoal, where the long (not the short) axes of the ellipses are parallel to the mean free surface.

In the southern hemisphere the tilt angle \( \alpha \) will have the opposite sign to that in the northern hemisphere, because the sign of the horizontal component of the Coriolis force changes when the latitude changes sign. The sign of the tilt angle \( \beta \), however, is independent of hemisphere.

Since it has been found that the Coriolis force on the particle velocity can easily be balanced at all times in most cases, by only a minor adjustment to the orbital parameters, when the waves have infinitesimal amplitude, then we conclude that in all probability this is what will take place, with the possible exception of transient or rapidly varying conditions. Therefore the general expectation is that there will be no torque caused by the Coriolis force that will change the wave angular momentum.

At finite amplitude the paradox of the Stokes drift remains to be discussed, i.e. how can the Coriolis force on the Stokes drift be balanced? Now possibly the Stokes drift itself could be the result of a slight imbalance in the cyclostrophic pair of forces. One way to think about it is to consider the gravity torque on long crested waves of finite amplitude (Kenyon, 1996). Even if the front and back faces of the wave remain symmetric at finite amplitude, as in the classical Stokes wave, the gravity torque will cause the horizontal pressure force to overbalance the centrifugal force on the front face of the wave and the centrifugal force to overbalance the pressure force on the back face. The net result is an acceleration of the particles in the direction of wave propagation, which would produce a net displacement of the particles in the direction of wave propagation after each wave period. This would then qualitatively and physically explain the existence of the Stokes drift, which was explained mathematically so many years ago by Stokes (1847).

The Coriolis force on the particle velocity could still be balanced at finite amplitude and at all times by the appropriate tilting of the orbital planes and change in shape of the orbit. Then the Coriolis force on the Stokes drift will automatically be taken care of in the process, that is it will be balanced by a small component of the centrifugal force. In other words balancing the Coriolis force on the Stokes drift is not a separate problem; it is all part of balancing the three forces (pressure, centrifugal, Coriolis) throughout a wave period as the particles perform their complete orbital motion, whether the amplitude is infinitesimal or finite. Also when the Coriolis force is in balance throughout the wave cycle, then it will not be possible for the waves to generate inertial motion.

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Appendix: Coriolis Torque

The general expression for the Coriolis torque can be worked out most easily using vector notation. Otherwise it is difficult to keep track of all the special cases that can occur.

Consider a progressive surface gravity wave at latitude $\theta$ propagating in the direction $\alpha$, measured counterclockwise from the east. The Coriolis torque at a fixed position is

$$\tau_C = \hat{r} \times \mathbf{F}$$  \hspace{1cm} (A1)

where $\hat{r}$ is the radius vector of an orbiting particle and $\mathbf{F}$ is the Coriolis force on that particle. The Coriolis force is

$$\mathbf{F} = -2\hat{\Omega} \times \mathbf{u}$$  \hspace{1cm} (A2)

where $\hat{\Omega}$ is the angular velocity of the Earth and $\mathbf{u}$ is the orbital velocity of a particle.

Now the angular velocity of the Earth can be represented as

$$\hat{\Omega} = \Omega (\hat{z} \sin \theta + \hat{y} \cos \theta)$$  \hspace{1cm} (A3)

where $\Omega$ is the angular speed of the Earth’s rotation and $\hat{z}$ and $\hat{y}$ are unit vectors in the vertical and north directions, respectively. Also the particle velocity can be expressed for circular orbital motion as

$$\mathbf{u} = \omega (\hat{l} \times \hat{r})$$  \hspace{1cm} (A4)

where $\omega$ is the wave frequency and $\hat{l}$ is a unit vector parallel to the angular momentum of the orbiting particle. Finally the radius vector needs to be stated in component form

$$\hat{r} = a (\hat{z} \cos \alpha + \hat{k} \sin \alpha)$$  \hspace{1cm} (A5)

where $a$ is the radius of a particle at the surface (wave amplitude), and $\hat{k}$ is a unit vector in the direction of wave propagation (i.e. in the wave number direction and perpendicular to $\hat{l}$).

With the help of equations (A2)–(A5), equation (A1) can be worked out in detail. For this purpose it is convenient to present the time mean and fluctuating parts of the Coriolis torque separately. The time mean part is

$$\bar{\tau}_C = \Omega \omega a^2 \left[ \hat{k} (\sin \theta) - \hat{z} (\cos \theta \sin \alpha) \right]$$  \hspace{1cm} (A6)

and the fluctuating part is

$$\tilde{\tau}_C = \Omega \omega a^2 \left[ \hat{k} (\cos \theta \sin \alpha \sin 2 \alpha + \sin \theta \cos 2 \alpha) \\ + \hat{z} (\cos \theta \sin \alpha \cos 2 \alpha - \sin \theta \sin 2 \alpha) \right]$$  \hspace{1cm} (A7)

and $\tau_C = \bar{\tau}_C + \tilde{\tau}_C$.

It can be seen from (A6) and (A7) that both the mean and fluctuating parts of the Coriolis torque have two components, a horizontal component directed parallel to the wave number, and a vertical component. Also (A7) shows that the frequency of the fluctuating Coriolis torque is double the wave frequency.

The tendency of the horizontal mean torque in (A6) is to rotate the angular momentum vectors clockwise in the northern hemisphere and counterclockwise in the southern hemisphere (the torque changes sign between hemispheres due to the latitude dependence). The vertical mean torque changes sign if the waves travel north or south but the sign is independent of hemisphere.

References


